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Unified laguerre-based poly-Apostol-type polynomials

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ABSTRACT

In this paper, we define and investigate the unified Laguerre-based poly-Apostol type polynomials. We obtain some identities and recurrence relations for these polynomials. Some symmetry identities and multiplication formula are also given.

1. Introduction

The generalized Apostol-Bernoulli polynomials $B_n^{(\alpha)}(x;\lambda)$ of order α are defined by Luo (2009) and Srivastava and Manocha (1984) through the generating relation:

$$\sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{t}{\lambda e^t-1}\right)^{(\alpha)} e^{xt}, |t+log\lambda|<2\pi;\ 1^{\alpha}=1,$$

where α and λ are the arbitrary real or complex parameters and $x\in R$.The Apostol-Bernoulli polynomials and the Apostol-Bernoulli numbers are given by

$$B_n(x;\lambda)=B_n^{(1)}(x;\lambda), B_n(\lambda)=B_n(0;\lambda), n\in N_0,$$

respectively. The case $\lambda=1$in the above relations give the classical Bernoulli polynomials $B_n(x)$ and the classical Bernoulli numbers B_n.

Recently, for the arbitrary real or complex parameters α, λ and $x\in R$, Luo (2009) generalized the Apostol-Euler polynomials $E_n^{(\alpha)}(x;\lambda)$ of order α by the generating relation

$$\sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{2}{\lambda e^t+1}\right)^{(\alpha)} e^{xt}, |t+log\lambda|<\pi;\ 1^{\alpha}=1.$$

The Apostol-Euler polynomials and the Apostol-Euler numbers are given by

$$E_n(x;\lambda)=E_n^{(1)}(x;\lambda), E_n(\lambda)=E_n(0;\lambda),$$

respectively. The above relations give the classical Euler polynomials $E_n(x)$ and the classical Euler number E_n when $\lambda=1$.

Let $x\in R$. For an arbitrary real or complex parameters α and λ, the Apostol-Genocchi polynomials of order α are defined by Luo (2009) and Srivastava and Manocha (1984)

$$\sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}=\left(\frac{2t}{\lambda e^t+1}\right)^{(\alpha)} e^{xt}, |t+log\lambda|<\pi;\ 1^{\alpha}=1.$$

The Apostol-Genocchi polynomials and Apostol-Genocchi numbers are given by

$$G_n(x;\lambda)=G_n^{(1)}(x;\lambda), G_n(\lambda)=G_n(0;\lambda),$$

respectively. When $\lambda=1$, the above relations give the classical Genocchi polynomials $G_n(x)$ and the classical Genocchi numbers G_n.

The two variable Laguerre polynomials $L_n(x;y)$ are defined by the generating functions (Dattoli and Torre, 1998)

$$\sum_{n=0}^{\infty} L_n(x;y)\frac{t^n}{n!}=e^{yt}C_0(xt) \quad (1)$$

where $C_0(x)$ is the 0-th order Tricomi function (Dattoli and Torre, 1998)

$$C_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{(r!)^2}. \quad (2)$$

From (1) and (2), we get

$$L_n(x,y)=\sum_s^n\binom{n}{s}\frac{n!(-1)^s x^s y^{n-s}}{(n-s)!(s!)^2}. \quad (3)$$

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The multiple power sums are defined by Luo (2009) as follows

$$S_k^{(l)}(m;\lambda)=\sum_{\substack{0\le v_1\le\cdots\le v_m=l\\ v_1+v_2+\cdots+v_m}}\binom{l}{v_1,v_2,\ldots,v_m}\lambda^{v_1+2v_2+\cdots+mv_m}(v_1+2v_2+\cdots+mv_m)^k. \quad (4)$$

From (4), we have (Luo, 2009).

$$\left(\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}\right)^l=\lambda^{(-l)}\sum_{n=0}^{\infty}\left\{\sum_p^n\binom{n}{p}(-l)^{n-p}S_p^{(l)}(m;\lambda)\right\}\frac{t^n}{n!} \quad (5)$$

From (5), for $l=1$

$$\frac{1-\lambda^m e^{mt}}{1-\lambda e^t}=\frac{1}{\lambda}\sum_{n=0}^{\infty}\left\{\sum_p^n\binom{n}{p}(-1)^{n-p}S_p^{(1)}(m;\lambda)\right\}\frac{t^n}{n!}. \quad (6)$$

The Stirling numbers of the second kind defined by Ozden et al. (2010) as

$$\sum_{n=0}^{\infty}S(n,v,a,b,\beta)\frac{t^n}{n!}=\frac{(\beta^b e^t-a^b)^v}{v!} \quad (7)$$

where $v,a,b,\beta\in R, a\neq b$.

Unified Apostol-Bernoulli, Euler and Genocchi polynomials are defined by Ozarslan (2011)

$$\sum_{n=0}^{\infty}P_{n,\beta}^{(\alpha)}(x;k;a,b)\frac{t^n}{n!}=\left(\frac{2^{1-k}t^k}{\beta^b e^t-a^b}\right)^{\alpha}e^{xt},$$
$$k\in N_0, a,b\in R\backslash\{0\}, \alpha,\beta\in C. \quad (8)$$

The 2-variable Kample de Feriet Hermite polynomials are defined in (Ozarslan, 2013; Pathan and Khan, 2014) as follows

$$\sum_{n=0}^{\infty}H_n^{(2)}(x,y)\frac{t^n}{n!}=e^{xt+yt^2}. \quad (9)$$

Definition 1: Let $\alpha\in N_0$, λ be an arbitrary real or complex parameter $x,y,z\in R$. The Laguerre-based generalized Apostol-Bernoulli polynomials are defined in Khan and Usman (2016) as following generating functions

$$\sum_{n=0}^{\infty}\left({}_L\mathrm{B}_n^{(\alpha)}(x,y,z;\lambda)\right)\frac{t^n}{n!}=\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{yt+zt^2}C_0(xt)$$
$$\{|t|<2\pi \text{ when } \alpha\in C, \lambda=1,\ |t|<|log\lambda| \text{ when } \alpha\in N_0, \lambda\neq 1, 1^{\alpha}=1\}. \quad (10)$$

For $k\in Z$, k>1 then k-th polylogarithm is defined by Bayad and Hamahata (2011) as

$$Li_k(z)=\sum_{n=1}^{\infty}\frac{z^n}{n!}. \quad (11)$$

This function is convergent for $|z|<1$, when k=1

$$Li_1(z)=-log(1-z).$$

Kim and Kim (2015) defined the poly-Bernoulli polynomials as

$$\sum_{n=0}^{\infty}B_n^{(k)}(x)\frac{t^n}{n!}=\frac{Li_k(1-e^{-t})}{e^t-1}e^{xt}. \quad (12)$$

Hamahata (2014) defined the poly-Euler polynomials by the following generating functions

$$\sum_{n=0}^{\infty}E_n^{(k)}(x)\frac{t^n}{n!}=\frac{2Li_k(1-e^{-t})}{t(e^t+1)}e^{xt}. \quad (13)$$

Kim et al. (2014) defined poly-Genocchi polynomials as

$$\sum_{n=0}^{\infty}G_n^{(k)}(x)\frac{t^n}{n!}=\frac{2Li_k(1-e^{-t})}{e^t+1}e^{xt}. \quad (14)$$

For k=1 in (12), (13) and (14), we get the classical Bernoulli, Euler and Genocchi polynomials respectively,

$$B_n^{(1)}(x)=B_n(x), E_n^{(1)}(x)=E_n(x), G_n^{(1)}(x)=G_n(x).$$

By the motivation of the definition of Khan and Usman (2016), we define the following expression.

Definition 2: We define unified Laguerre-based poly-Apostol type polynomials as

$$\sum_{n=0}^{\infty}\left({}_LP_{n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{t^n}{n!}=\left(\frac{2^{1-l}\left(Li_k(1-e^{-t})\right)^l}{\beta^b e^t-a^b}\right)^{\alpha}e^{yt+zt^2}C_0(xt)$$
$$l,k\in N_0, a,b>0, a,b\in R\backslash\{0\}, \alpha,\beta\in C. \quad (15)$$

For the existence of the expansion, we need

i. $|t|<2\pi$ when $\alpha\in N_0, k=1$ and $\left(\frac{\beta}{\alpha}\right)^b=1, |t|<2\pi$ when $\alpha\in N_0, k=1,2,3$ and $\left(\frac{\beta}{\alpha}\right)^b=1, |t|<\left|blog\left(\frac{\beta}{\alpha}\right)\right|$, when $\alpha\in N_0, k\in N$ and $\left(\frac{\beta}{\alpha}\right)^b\neq 1, 1^{\alpha}:=1, a,b\in C\backslash\{0\}, \beta\in C$.

ii. $|t|<\pi$ when $\left(\frac{\beta}{\alpha}\right)^b=-1, |t|<\left|blog\left(\frac{\beta}{\alpha}\right)\right|$ when $\left(\frac{\beta}{\alpha}\right)^b\neq 1, k=0, 1^{\alpha}:=1, a,b\in C\backslash\{0\}, \alpha,\beta\in C$.

iii. $|t|<\pi$ when $\alpha\in N_0$ and $\left(\frac{\beta}{\alpha}\right)^b=-1, x,y,z\in R, k\in N, \beta\in C, a,b\in C\backslash\{0\}, 1^{\alpha}:=1$ where $w=|w|e^{i\theta}, -\pi<|\theta|<\pi$ and $logw=log|w|+i\theta$.

Remark 1: Setting k=l=1, a=b=1, z=0 and $\beta=\lambda$ in (15), we have Laguerre-based Apostol-Bernoulli polynomials

$$\left(\frac{t}{\lambda e^t-1}\right)^{\alpha}e^{yt}C_0(xt)=\sum_{n=0}^{\infty}\left({}_LP_{n,\lambda}^{[1,\alpha]}(x,y,0;1,1,1,1)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left({}_LB_n^{(\alpha)}(x,y,0;\lambda)\right)\frac{t^n}{n!}. \quad (16)$$

Remark 2: Choosing k=1, l=0, a=-1, b=1, z=0 and $\beta=\lambda$ in (15), we get Laguerre-based Apostol-Euler polynomials

$$\left(\frac{2}{\lambda e^t+1}\right)^{\alpha}e^{yt}C_0(xt)=\sum_{n=0}^{\infty}\left({}_LP_{n,\lambda}^{[1,\alpha]}(x,y,0;0,1,-1,1)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left({}_LE_n^{(\alpha)}(x,y,0;\lambda)\right)\frac{t^n}{n!}. \quad (17)$$

Remark 3: Putting k=1, l=1, a=$-\frac{1}{2}$, b=1, z=0 and $\beta=\lambda$ in (15), we get Laguerre-based Apostol-Genocchi polynomials

$$\left(\frac{2t}{\lambda e^t+1}\right)^\alpha e^{yt}C_0(xt) = \sum_{n=0}^{\infty}\left({}_LP_{n,\lambda/2}^{[1,\alpha]}\left(x,y,0;1,1,-\frac{1}{2},1\right)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left({}_LG_n^{(\alpha)}(x,y,0;\lambda)\right)\frac{t^n}{n!}. \quad (18)$$

Laguerre −based Apostol-Bernoulli, Laguerre-based Apostol-Euler and Laguerre-based Apostol-Genocchi polynomials are studied and investigated by Khan and Usman (2016). Luo (2009), Luo and Srivastava (2005) and Srivastava (2011) introduced the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials and proved some theorems and relations for these polynomials. Kurt (2016a, 2016b) introduced the unified family of generalized Apostol-type polynomials and gave some symmetry identities and recurrences relations for these polynomials. Ozden et al. (2010) introduced the unified representation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. Ozarslan (2011) studied the unified Apostol-Bernoulli, Euler and Genocchi polynomials. He gave some theorems for the Hermite-based unified Apostol-Bernoulli, Euler and Genocchi polynomials.

Hamahata (2014) and Bayad and Hamahata (2011) defined and investigated poly-Bernoulli polynomials. Kim and Kim (2015) gave some recurrence relation for the higher-order poly-Bernoulli polynomials. Kim et al. (2014) introduced poly-Genocchi polynomials. Pathan and Khan (2016, 2015, 2014) introduced the Hermite-based Bernoulli polynomials, Euler polynomials and gave some relation for these polynomials. Khan and Usman (2016) introduced a new class of Laguerre-based generalized Apostol polynomials. He also gave some symmetric relations for these polynomials.

In this work, we define unified Laguerre-based poly-Apostol type polynomials. After we give some implicit relations for these polynomials. Also we prove some symmetric relations for the unified Laguerre-based poly-Apostol type polynomials.

2. Some ımplicit relations for the unified laguerre-based poly-Apostol type polynomials

In this section, we will give some relations between 2-variable Hermite polynomials and the unified Laguerre-based poly-Apostol type polynomials. Also, we will give some implicit relation for these polynomials.

Theorem 1: There is the following relations between unified Laguerre-based poly-Apostol type polynomials and two variable Hermite polynomials

$H_n^{(2)}(x,y)$ as

$${}_LP_{n,\beta}^{[k,\alpha]}(x,y+u,z+v;l,k,a,b)$$
$$= \sum_s^r\binom{r}{s}\ {}_LP_{n-s,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)H_s^{(2)}(u,v). \quad (19)$$

Proof: From (9) and (15)

$$\sum_n^\infty\ {}_LP_{n,\beta}^{[k,\alpha]}(x,y+u,z+v;l,k,a,b)\frac{t^n}{n!}$$
$$= \left(\frac{2^{1-l}\left(Li_k(1-e^{-t})\right)^l}{\beta^b e^t-a^b}\right)^\alpha e^{(y+u)t+(z+v)t^2}C_0(xt)$$
$$= \sum_n^\infty\ {}_LP_{n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^n}{n!}\sum_s^\infty H_s^{(2)}(u,v)\frac{t^s}{s!}.$$

By using Cauchy product, equating the coefficients of $\frac{t^n}{n!}$, we have (19).

Theorem 2: The unified Laguerre-based poly-Apostol type polynomials satsify the following equation

$${}_LP_{n,\beta}^{[k,1]}(x,y,z+v;l,k,a,b) =$$
$$\sum_{l=0}^{\left[\frac{n}{2}\right]}\frac{n!}{l!(n-2l)!}\left(\ {}_LP_{n-2l,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right)v^l. \quad (20)$$

Proof: For $\alpha = 1$, from (15)

$$\sum_n^\infty\ {}_LP_{n,\beta}^{[k,1]}(x,y,z+v;l,k,a,b)\frac{t^n}{n!} = \frac{2^{1-l}\left(Li_k(1-e^{-t})\right)^l}{\beta^b e^t-a^b}e^{yt+zt^2}C_0(xt)e^{vt^2}$$
$$= \sum_n^\infty\ {}_LP_{n,\beta}^{[k,1]}(x,y,z;l,k,a,b)\frac{t^n}{n!}\sum_n^\infty\frac{v^n t^{2n}}{n!}.$$

Using Cauchy product, comparing the coefficients both sides, we get (20).

Theorem 3: The following relation holds

$${}_LP_{n,\beta}^{[k,\alpha]}(0,y,z;l,k,a,b)$$
$$= \sum_m^n\binom{n}{m}\ {}_LP_{m,\beta}^{[k,\alpha]}(l,k,a,b)H_{n-m}^{(2)}(y,z). \quad (21)$$

(21) can be obtain easily from (9) and (15).

Theorem 4: Let $a,b > 0, a \neq b, x,y,z \in R$. There is the following relation between unified poly-Apostol type polynomials and Laguerre polynomials

$${}_LP_{n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b) =$$
$$n!\sum_{j=0}^{\left[\frac{n}{2}\right]}\sum_m^{n-2j}\binom{n}{m}\frac{{}_LP_{m,\beta}^{[k,\alpha]}(l,k,a,b)L_{n-2j-m}(x,y)}{(n-2j)!j!}z^j. \quad (22)$$

The proof of (22) can be obtain from (1) and (15).

Theorem 5: Unified Laguerre-based poly-Apostol type polynomials satisfy the following relation

$$\sum_p^n\binom{n}{p}\sum_q^m\binom{m}{q}(v-y)^{m+n-r-s}\ {}_LP_{m+n-p-q,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b) =$$
$${}_LP_{m+n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b). \quad (23)$$

Proof: We replace *t* by *t+u* and rewrite the generating function (15) as

$$\left(\frac{2^{1-l}\left(Li_k(1-e^{-(t+u)})\right)^l}{\beta^b e^{(t+u)}-a^b}\right)^\alpha e^{z(t+u)^2}C_0(x+(t+u))$$
$$= e^{-y(t+u)}\sum_{m,n=0}^\infty\ {}_LP_{m+n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}. \quad (24)$$

Replacing y by u in the above equation and equating the resulting equation to the above equation. We get

$$e^{(v-y)(t+u)}\sum_{m,n=0}^{\infty}\ {}_LP_{m+n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^n u^m}{n!\ m!}$$
$$=\sum_{m,n=0}^{\infty}\ {}_LP_{m+n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}. \qquad (25)$$

On expanding exponential function (25) gives

$$\sum_{N=0}^{\infty}\frac{[(v-y)(t+u)]^N}{N!}\sum_{m,n=0}^{\infty}\ {}_LP_{m+n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$=\sum_{m,n=0}^{\infty}\ {}_LP_{m+n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^n}{n!}\frac{u^m}{m!} \qquad (26)$$

which on using formula (Srivastava, 2011).

$$\sum_{N=0}^{\infty}\frac{f(N)(x+y)^N}{N!}=\sum_{m,n=0}^{\infty}f(m+n)\frac{x^n}{n!}\frac{y^m}{m!} \qquad (27)$$

in the left hand side becomes

$$\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}(v-y)^{p+q}\frac{t^p}{p!}\frac{u^q}{q!}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\ {}_LP_{r+s,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\frac{t^r}{r!}\frac{u^s}{s!}.$$

From here, we get

$$\sum_{m}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\binom{n}{p}\sum_{q=0}^{m}\binom{m}{q}(v-y)^{m+n-r-s}\left(\ {}_LP_{m+n-p-q,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\ {}_LP_{m+n-p-q,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{t^n}{n!}\frac{u^m}{m!}.$$

Comparing of the coefficients of both sides, we have (23).

Remark 4: For $k=l=1, a=b=1, z=0, \beta=\lambda$ in (23), we have

$$\sum_{p=0}^{n}\binom{n}{p}\sum_{q=0}^{m}\binom{m}{q}(v-y)^{m+n-r-s}\left(\ {}_LB_{m+n-p-q}^{(\alpha)}(x,y;\lambda)\right)$$
$$=\left(\ {}_LB_{m+n}^{(\alpha)}(x,v,z;\lambda)\right).$$

Remark 5: For $k=1,\ l=0, a=-1,\ z=0, \beta=\lambda$ in (23), we have

$$\sum_{p=0}^{n}\binom{n}{p}\sum_{q=0}^{m}\binom{m}{q}(v-y)^{m+n-r-s}\left(\ {}_LE_{m+n-p-q}^{(\alpha)}(x,y;\lambda)\right)$$
$$=\left(\ {}_LE_{m+n}^{(\alpha)}(x,v,z;\lambda)\right).$$

Remark 6: For $k=l=b=1, a=-\frac{1}{2},\ z=0, \beta=\frac{\lambda}{2}$ in (23), we have

$$\sum_{p=0}^{n}\binom{n}{p}\sum_{q=0}^{m}\binom{m}{q}(v-y)^{m+n-r-s}\left(\ {}_LG_{m+n-p-q}^{(\alpha)}(x,y;\lambda)\right)$$
$$=\left(\ {}_LG_{m+n}^{(\alpha)}(x,v,z;\lambda)\right).$$

3. Some symmetry ıdentitites for the unified laguerre-based poly-Apostol type polynomials

In this section, we give some symmetric identities for the unified Laguerre-based poly-Apostol type polynomials. Also, we prove some relation between these polynomials and the Stirling numbers of the second kind. Further, we give the multiple Powers sums for the unified Laguerre-based poly-Apostol type polynomials.

Theorem 6: Unified Laguerre-based poly-Apostol type polynomials satisfy the following symmetry identities

$$\sum_{m=0}^{n}\binom{n}{m}\left(\ {}_LP_{n-m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)$$
$$\times c^{n-m}d^n\left(\ {}_LP_{m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\left(\ {}_LP_{n-m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)$$
$$\times d^{n-m}c^n\left(\ {}_LP_{m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right). \qquad (28)$$

Proof: Let

$$f(t)=\left(\frac{c^l d^l 2^{2(1-l)}\left(Li_k(1-e^{-t})\right)^{2l}}{(\beta^b e^{ct}-a^b)(\beta^b e^{dt}-a^b)}\right)^{\alpha}$$
$$\times e^{(d+c)yt+(d^2+c^2)zt^2}C_0(xdt)C_0(xct)$$
$$=\left(\frac{2^{(1-l)}\left(cLi_k(1-e^{-t})\right)^{l}}{(\beta^b e^{ct}-a^b)}\right)^{\alpha}e^{cyt+c^2zt^2}C_0(xct)$$
$$\times\left(\frac{2^{(1-l)}\left(dLi_k(1-e^{-t})\right)^{l}}{(\beta^b e^{dt}-a^b)}\right)^{\alpha}e^{dyt+d^2zt^2}C_0(xdt)$$
$$=\sum_{n=0}^{\infty}\left(\ {}_LP_{n,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{c^n t^n}{n!}$$
$$\times\sum_{m=0}^{\infty}\left(\ {}_LP_{m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{d^m t^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\left(\ {}_LP_{n-m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)$$
$$\times c^{n-m}d^n\left(\ {}_LP_{m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{t^n}{n!}. \qquad (29)$$

In similar manner

$$f(t)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\left(\ {}_LP_{n-m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)$$
$$\times d^{n-m}c^n\left(\ {}_LP_{m,\beta}^{[k,\alpha]}(x,y,z;l,k,a,b)\right)\frac{t^n}{n!}. \qquad (30)$$

From (29) and (30), we obtain (28).

Theorem 7: There is the following relation between the unified Laguerre-based Apostol type polynomials and Stirling number of the second kind

$$\sum_{q=0}^{n}\binom{n}{q}\left(\ {}_LP_{n-q,\beta}^{[1,1]}(x,y,z;l,1,a,b)\right)S(q,1,a,b,\beta)$$
$$=2^{(1-l)}\sum_{r=0}^{n-l}\frac{n!H_{n-r-l}^{(2)}(y,z)(-1)^r x^r}{(n-l-r)!r!}. \qquad (31)$$

Proof: From (15), for $\alpha=k=1$. We write as

$$\sum_{n=0}^{\infty}\left(\ {}_LP_{n,\beta}^{[1,1]}(x,y,z;l,1,a,b)\right)\frac{t^n}{n!}(\beta^b e^t-a^b)=$$
$$2^{1-l}t^l e^{yt+zt^2}C_0(xt)$$
$$\sum_{n=0}^{\infty}\left(\ {}_LP_{n,\beta}^{[1,1]}(x,y,z;l,1,a,b)\right)\frac{t^n}{n!}\alpha!$$
$$\times\sum_{n=0}^{\infty}S(n,1,a,b,\beta)\frac{t^n}{n!}$$
$$=2^{1-l}t^l\sum_{n=0}^{\infty}\sum_{r=0}^{n}n!\frac{H_{n-r}^{(2)}(y,z)(-1)^r x^r}{(n-r)!(r!)^2}\frac{t^n}{n!}.$$

By using Cauchy product and since

$$\left(\ {}_LP_{-q,\beta}^{[1,1]}(x,y,z;l,1,a,b)\right)=\cdots=$$
$$\left(\ {}_LP_{n-(l-1),\beta}^{[1,1]}(x,y,z;l,1,a,b)\right)=0.$$

We obtain (31).

Theorem 8: The following relation holds

$$\sum_{r=0}^{n}\binom{n}{r}\left(\ {}_LP_{r,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right)c^r d^l a^{b(d-1)}$$

$$\times \sum_{m=0}^{c-1} \left(\frac{\beta}{a}\right)^m (md)^{n-r}$$
$$= \sum_{r=0}^{n} \binom{n}{r} \left({}_{L}P_{r,\beta}^{[k,1]} \left(\frac{x}{d}, \frac{cy}{d}, \frac{c^2 z}{d^2}; l, k, a, b\right)\right) d^r c^l a^{b(d-1)}$$
$$\times \sum_{m=0}^{d-1} \left(\frac{\beta}{a}\right)^m (mc)^{n-r}. \tag{32}$$

Proof: From (5), for $\alpha = 1$. Let

$$g(t) = \frac{2^{1-l} c^l d^2 \left(Li_k(1-e^{-t})\right)^l (\beta^{bd} e^{cdt} - a^{bd})}{(\beta^b e^{ct} - a^b)(\beta^b e^{dt} - a^b)} e^{cyt + c^2 z t^2} C_0(xct)$$
$$= \frac{2^{1-l}\left(cLi_k(1-e^{-t})\right)^l}{(\beta^b e^{ct} - a^b)} e^{cyt + c^2 z t^2} C_0(xct) \frac{d^l a^{bd}\left(1-\left(\frac{\beta}{a}\right)^{bd} e^{cdt}\right)}{a^b\left(1-\left(\frac{\beta}{a}\right)^b e^{dt}\right)}$$
$$= \sum_{n=0}^{\infty} \left({}_{L}P_{n,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right) \frac{c^n t^n}{n!} d^l a^{b(d-1)}$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{c-1} \left(\frac{\beta}{a}\right)^m (md)^n \frac{t^n}{n!}.$$

By using the Cauchy product, we have

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \left({}_{L}P_{r,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right) c^r d^l a^{b(d-1)}$$
$$\times \sum_{m=0}^{c-1} \left(\frac{\beta}{a}\right)^m (md)^{n-r} \frac{t^n}{n!}. \tag{33}$$

Similiarly

$$g(t) = \frac{2^{1-l}\left(dLi_k(1-e^{-t})\right)^l}{(\beta^b e^{dt} - a^b)} e^{\frac{c}{d} y dt + \frac{c^2}{d^2} z d^2 t^2} C_0\left(\frac{x}{d} dt\right) c^l \frac{(\beta^{bd} e^{cdt} - a^{bd})}{(\beta^b e^{ct} - a^b)}$$
$$= \sum_{n=0}^{\infty} {}_{L}P_{n,\beta}^{[k,1]} \left(\frac{x}{d}, \frac{cy}{d}, \frac{c^2 z}{d^2}; l, k, a, b\right) \frac{d^n t^n}{n!} c^l a^{b(d-1)}$$
$$\times \sum_{n=0}^{\infty} \sum_{m=0}^{d-1} \left(\frac{\beta}{a}\right)^m (mc)^n \frac{t^n}{n!}.$$

Using Cauchy product, we get

$$\sum_{n=0}^{\infty} \sum_{r=0}^{n} \binom{n}{r} \left({}_{L}P_{r,\beta}^{[k,1]} \left(\frac{x}{d}, \frac{cy}{d}, \frac{c^2 z}{d^2}; l, k, a, b\right)\right) d^r c^l a^{b(d-1)}$$
$$\times \sum_{m=0}^{d-1} \left(\frac{\beta}{a}\right)^m (mc)^{n-r} \frac{t^n}{n!}. \tag{34}$$

Equating the coeffiecents of $\frac{t^n}{n!}$ both sides of the equations (33) and (34), we obtain (32).

Theorem 9: There is the following symmetric relations between multiple power sums and unified Laguerre-based poly-Apostol polynomials

$$d^{(\alpha-1)l+2} \sum_{\gamma=0}^{n} \binom{n}{\gamma} \left({}_{L}P_{n-\gamma,\beta}^{[k,\alpha+1]}(x,y,z;l,k,a,b)\right) c^{n+1-\gamma}$$
$$\times \sum_{p=0}^{\gamma} \binom{\gamma}{p} \sum_{r=0}^{p} \binom{p}{r} (-\alpha)^{p-r} S_r^{(\alpha)}\left(c, \left(\frac{\beta}{a}\right)^b\right)$$
$$\times \left({}_{L}P_{\gamma-p,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right) d^\gamma$$
$$= c^{(\alpha-1)l+2} \sum_{\gamma=0}^{n} \binom{n}{\gamma} \left({}_{L}P_{n-\gamma,\beta}^{[k,\alpha+1]}(x,y,z;l,k,a,b)\right) d^{n+1-\gamma}$$
$$\times \sum_{p=0}^{\gamma} \binom{\gamma}{p} \sum_{r=0}^{p} \binom{p}{r} (-\alpha)^{p-r} S_r^{(\alpha)}\left(d, \left(\frac{\beta}{a}\right)^b\right)$$
$$\times \left({}_{L}P_{\gamma-p,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right) c^\gamma. \tag{35}$$

Proof: Let

$$h(t) = \frac{\left(2^{1-l} d^l c^l \left(Li_k(1-e^{-t})\right)^l\right)^{\alpha+2}}{(\beta^b e^{ct} - a^b)^{\alpha+1}} \frac{(\beta^{bd} e^{cdt} - a^{bd})^\alpha}{(\beta^b e^{dt} - a^b)^{\alpha+1}} e^{(d+c)yt + (d^2+c^2) z t^2}$$
$$\times C_0(xct) C_0(xdt)$$
$$= \left(\frac{2^{1-l}\left(cLi_k(1-e^{-t})\right)^l}{\beta^b e^{ct} - a^b}\right)^{\alpha+1} e^{cyt + c^2 z t^2} C_0(xct) c d^{(\alpha-1)l+2}$$
$$\times \left(\frac{\beta^{bd} e^{cdt} - a^{bd}}{\beta^b e^{dt} - a^b}\right)^\alpha \frac{2^{1-l}\left(dLi_k(1-e^{-t})\right)^l}{\beta^b e^{dt} - a^b} e^{dyt + d^2 z t^2} C_0(xdt).$$

By using (6) and Cauchy product, we have

$$= \sum_{n=0}^{\infty} \{ d^{(\alpha-1)l+2} \sum_{\gamma=0}^{n} \binom{n}{\gamma} \left({}_{L}P_{n-\gamma,\beta}^{[k,\alpha+1]}(x,y,z;l,k,a,b)\right)$$
$$\times c^{n+1-\gamma} \sum_{p=0}^{\gamma} \binom{\gamma}{p} \sum_{r=0}^{p} \binom{p}{r} (-\alpha)^{p-r} S_r^{(\alpha)}\left(c, \left(\frac{\beta}{a}\right)^b\right)$$
$$\times \left({}_{L}P_{\gamma-p,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right) d^\gamma \} \frac{t^n}{n!}. \tag{36}$$

In similiar manner,

$$h(t) =$$
$$\sum_{n=0}^{\infty} \{ c^{(\alpha-1)l+2} \sum_{\gamma=0}^{n} \binom{n}{\gamma} \left({}_{L}P_{n-\gamma,\beta}^{[k,\alpha+1]}(x,y,z;l,k,a,b)\right)$$
$$\times d^{n+1-\gamma} \sum_{p=0}^{\gamma} \binom{\gamma}{p} \sum_{r=0}^{p} \binom{p}{r} (-\alpha)^{p-r} S_r^{(\alpha)}\left(d, \left(\frac{\beta}{a}\right)^b\right)$$
$$\times \left({}_{L}P_{\gamma-p,\beta}^{[k,1]}(x,y,z;l,k,a,b)\right) c^\gamma \} \frac{t^n}{n!}. \tag{37}$$

Comparing the coeffiecents of $\frac{t^n}{n!}$ both sides of the equations (36) and (37). We obtain (35).

4. Conclusion

In this work, we apply the poly-Bernoulli polynomials to unified Laguerre-based polynomials. We obtain some symmetric identities and some relations fort he unified Laguerre-based Apostol type polynomials.

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